First-order logic with reachability for infinite-state systems

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Goal: decidability frontier of FO[R]

$$S_A \models \varphi$$

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Automaton:
finite control + storage

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Reachability structure:
infinite graph of configurations + $\to^*$

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FO[R] first-order with reachability:
middle ground between FO and MSO

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infinite graph of configurations + $\to^*$

$$S_A \models \varphi$$

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FO[R] first-order with reachability:
middle ground between FO and MSO

Decidable		**Undecidable**
1-stack automata	?	2-stacks automata
2-dimension VASS		VASS (aka Petri nets)

Key Question

Which features of storage mechanisms determine the decidability of FO[R]?

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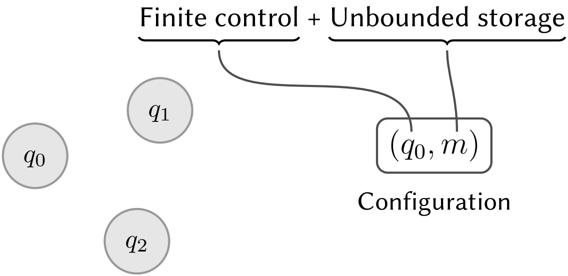
Main Result

We found a simple condition characterising storage mechanisms with decidable FO[R].

Valence Automata

Finite control + Unbounded storage

Valence Automata



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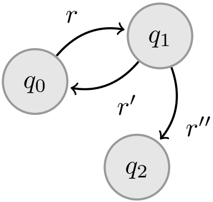
Finite control + Unbounded storage



$$(q_0, m) \longrightarrow (q_1, m \odot r)$$

Apply action

Valence Automata

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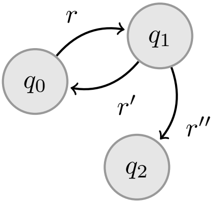


$$(q_0, m) \longrightarrow (q_1, m \odot r)$$

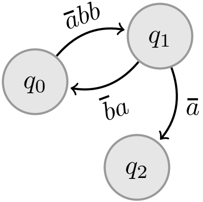
Apply action

Storage = Monoid $(M, \odot, \mathbf{1})$

- $m, r \in M$ set of storage contents and actions
- valid storage contents are the right-invertible elements:
  $$m \in \mathcal{R}_1(M) := \{x \in M \mid \exists r \in M : x \odot r = \mathbf{1}\}$$
- $\mathbf{1}$ is empty storage and no-op action

Valence Automata

Pushdown systems

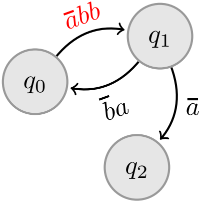


(q_0, ba)

Stack = Monoid $\left(\{a, b, \bar{a}, \bar{b}\}^*, \odot, \varepsilon\right)$

- a, b are push actions, $\bar{a}, \bar{b}$ are pop actions

Pushdown systems



$$(q_0, ba) \longrightarrow (q_1, ba \odot \bar{a}bb)$$

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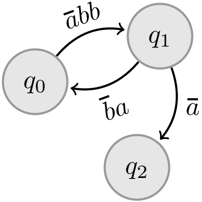

$$(q_0, ba) \quad \longrightarrow \quad (q_1, b\not{a} \odot \not{\bar{a}}bb)$$

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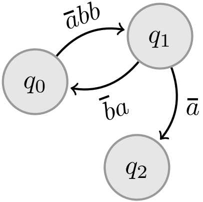

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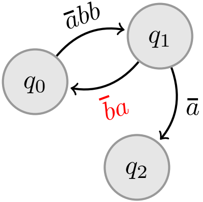

$$(q_0, ba) \longrightarrow (q_1, \quad bbb \quad)$$
$$\downarrow$$
$$(q_0, bbb \odot \bar{b}a)$$

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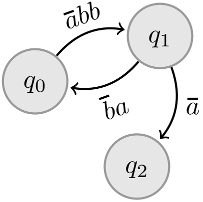

$$(q_0, ba) \longrightarrow (q_1, bbb)$$
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$$(q_0, bba)$$

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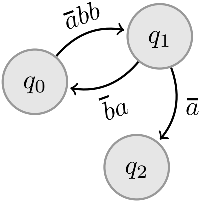

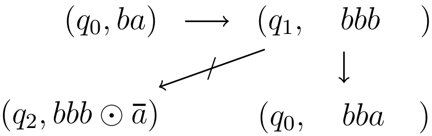

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$$\mathcal{R}_1(M) = \{a, b\}^*$$

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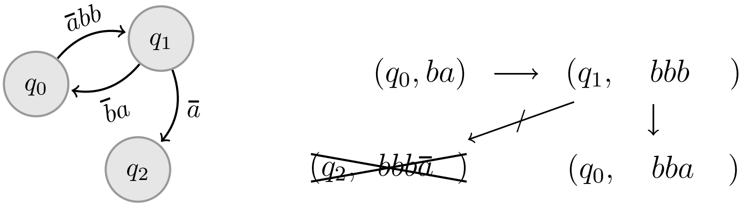

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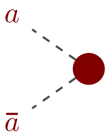
Graph monoids

- Graph $\Gamma = (V, E)$

Graph monoids



$a\bar{a} \equiv \varepsilon$

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- Generators $X_\Gamma = \{a_v, \bar{a}_v \mid v \in V\}$

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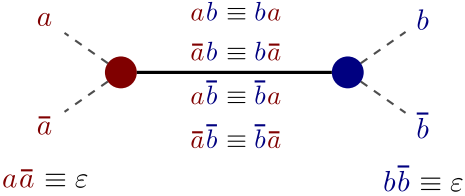

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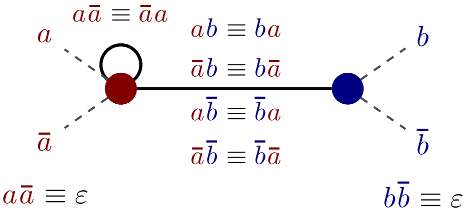

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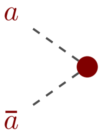
Examples of Graph Monoids

A partially blind counter: $\mathbb{B}$



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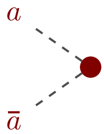

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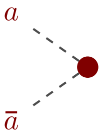

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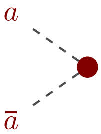

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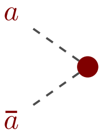

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$$a$$

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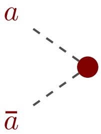

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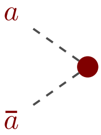

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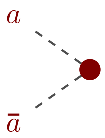

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$$\cancel{a}\cancel{\bar{a}}\bar{a} \equiv \bar{a}$$ not right-invertible!

Can only represent positive integers

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A blind counter: $\mathbb{Z}$

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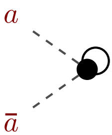

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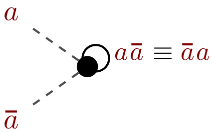

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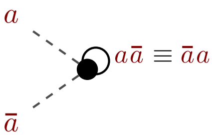

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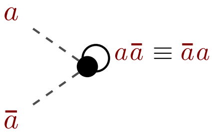

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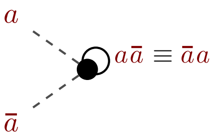

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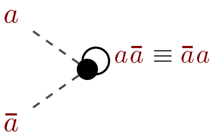

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$$a \cdot \bar{a}\bar{a}$$

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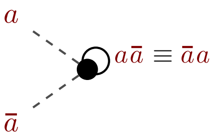

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now right-invertible!

($\bar{a}a \equiv a\bar{a} \equiv \varepsilon$)

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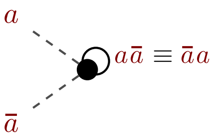

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Can also represent negative integers

Examples of Graph Monoids

A stack of two symbols: $\mathbb{B} * \mathbb{B}$

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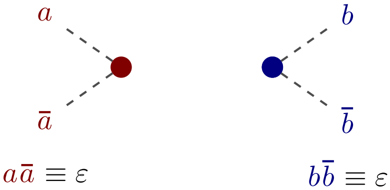

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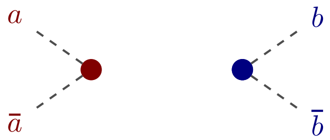

$a\bar{a} \equiv \varepsilon$ $b\bar{b} \equiv \varepsilon$

$aaab$

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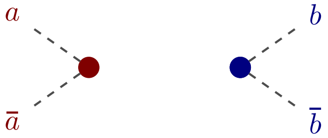

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$$aaab \cdot \bar{b}\bar{a}b$$

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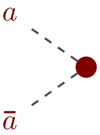

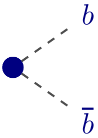

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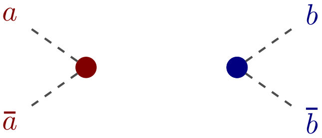

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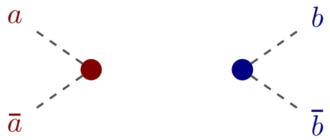

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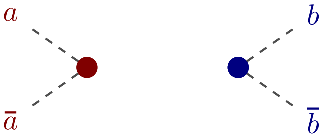

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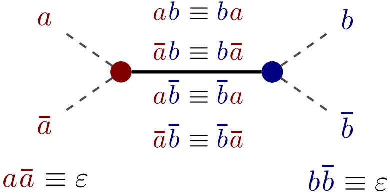

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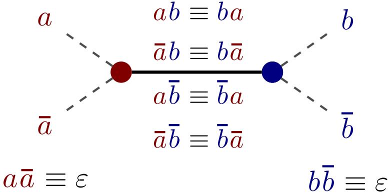

$$abaaba$$

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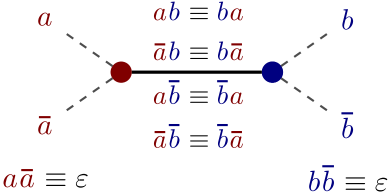

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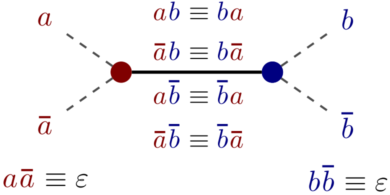

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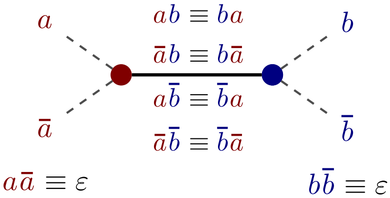

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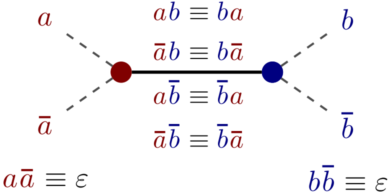

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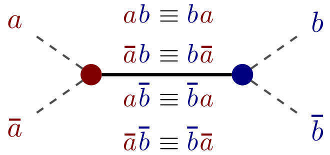

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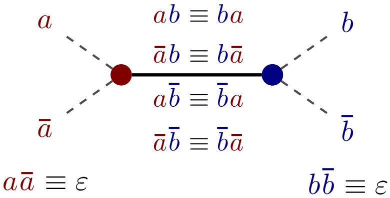
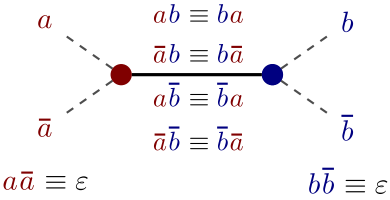


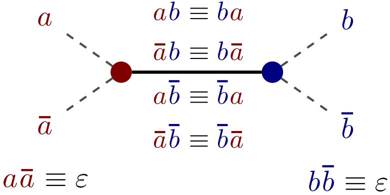
$$aaaabb\bar{a}$$

Examples of Graph Monoids



$$aaaab\bar{a}b$$

Examples of Graph Monoids



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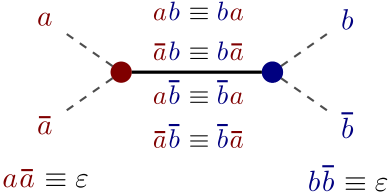
Examples of Graph Monoids



$$aaa\not{a}\not{\bar{a}}bb$$

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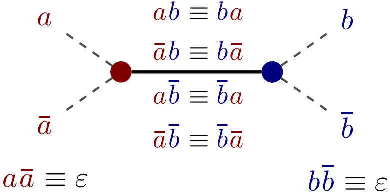

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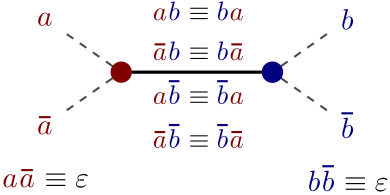
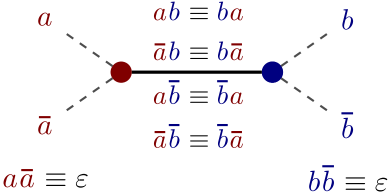


$$aaabb \cdot \bar{b}$$

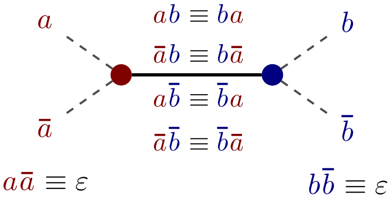
Examples of Graph Monoids



$$aaab\cancel{b}\cancel{\bar{b}}$$

Examples of Graph Monoids

Two partially blind counters: $\mathbb{B} \times \mathbb{B}$



$$aaab$$

Direct product

This works in general: $\mathbb{M}\Gamma_1 \times \mathbb{M}\Gamma_2 = \mathbb{M}\Gamma_{1,2}$:

$\mathbb{Z}$

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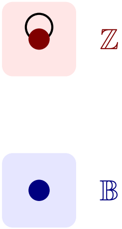

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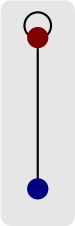

$\mathbb{Z} \times \mathbb{B}$

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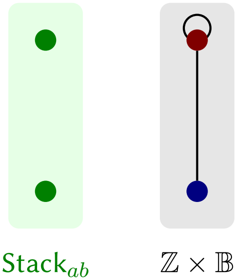

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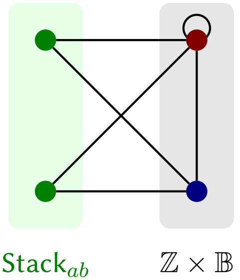

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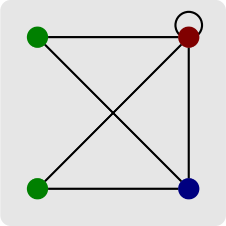

$\mathsf{Stack}_{ab} \times \mathbb{Z} \times \mathbb{B}$

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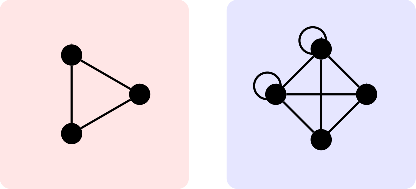
Free product: $\mathbb{M}\Gamma_1 * \mathbb{M}\Gamma_2 = \mathbb{M}(\Gamma_1 \uplus \Gamma_2)$.



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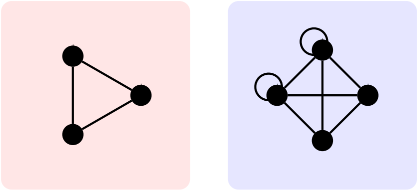

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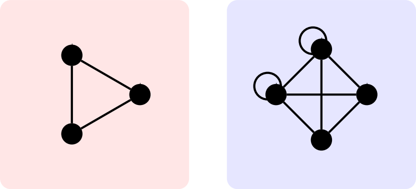

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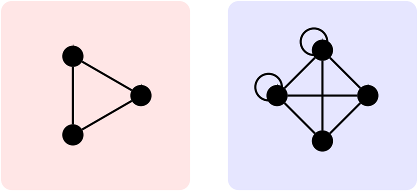

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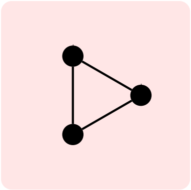

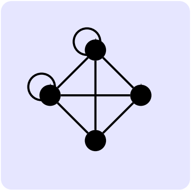

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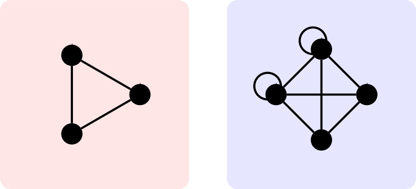

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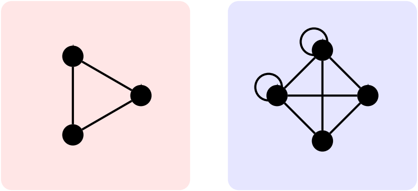

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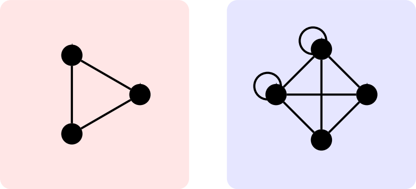

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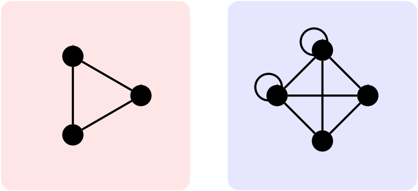
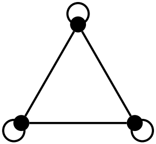
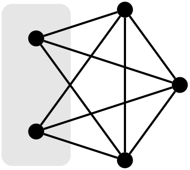


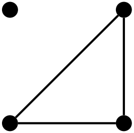
Graph monoids



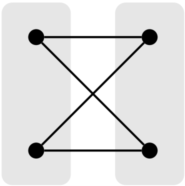
$\mathbb{Z}^3$



$(\mathbb{B} * \mathbb{B}) \times \mathbb{B}^3$

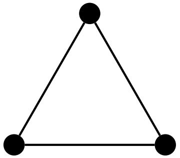


$\mathbb{B} * \mathbb{B}^3$

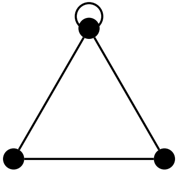


$(\mathbb{B} * \mathbb{B}) \times (\mathbb{B} * \mathbb{B})$

$\mathbb{B}^2$-triangles



$\mathbb{B}^2 \times \mathbb{B}$



$\mathbb{B}^2 \times \mathbb{Z}$

Def A graph is $\mathbb{B}^2$*-triangle-free* if it does not contain a $\mathbb{B}^2$-triangle as induced subgraph.

Main Result

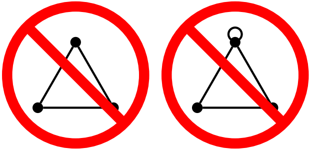
Theorem

FO[R] for valence systems over $\mathbb{M}\Gamma$ is decidable

if and only if

Γ is a disjoint union of $\mathbb{B}^2$-triangle-free cliques.

Characterisation of decidable cases



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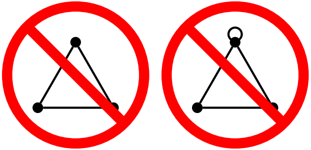
Allowed cliques:

Characterisation of decidable cases



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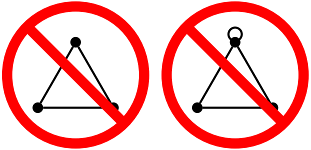

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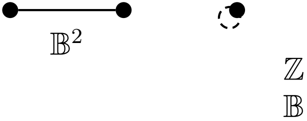

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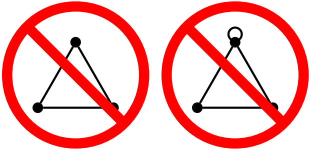

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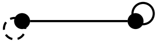

$\mathbb{Z} \times \mathbb{Z}$

$\mathbb{B} \times \mathbb{Z}$

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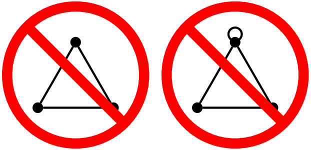

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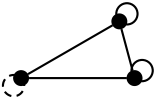
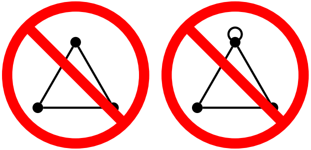


$\mathbb{Z} \times \mathbb{Z}^2$
$\mathbb{B} \times \mathbb{Z}^2$

Characterisation of decidable cases



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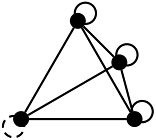

$\mathbb{Z} \times \mathbb{Z}^3$
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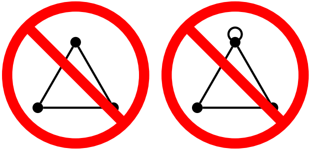

Γ is a disjoint union of $\mathbb{B}^2$-triangle-free cliques

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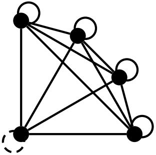
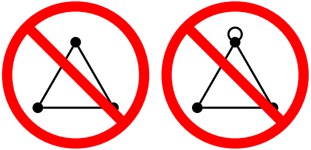


$\mathbb{Z} \times \mathbb{Z}^4$
$\mathbb{B} \times \mathbb{Z}^4$

Characterisation of decidable cases



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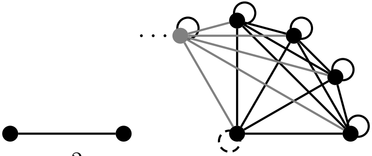


$\mathbb{B}^2$

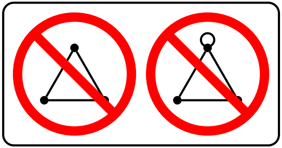
$\mathbb{Z} \times \mathbb{Z}^n$
$\mathbb{B} \times \mathbb{Z}^n$

$\mathbb{M}\Gamma_1 * \mathbb{M}\Gamma_2 * \cdots * \mathbb{M}\Gamma_k$ where Γ_i are cliques as above.

Operationally: Stack with as entries either

- 2 partially blind counters, or
- a partially blind counter with n blind counters.

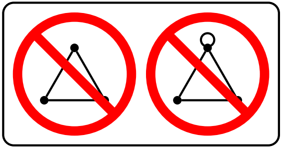
Decidability

 $\Longrightarrow$ FO[R] is decidable

Proof By showing **automaticity of the reachability structure**: the step and reachability relations can be represented by *regular relations*.

By (Khoussainov & Nerode 1995) the first-order theory of an automatic structure is decidable.

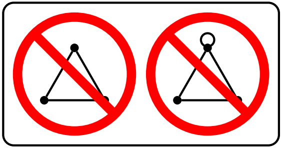
Decidability

 $\Longrightarrow$ FO[R] is decidable

We show automaticity for the reachability structures over:

- $\mathbb{B} \times \mathbb{B}$ a consequence of Presburger definability of reachability for 2-dimension VASS (Leroux & Sutre 2004)
- $\mathbb{B} \times \mathbb{Z}^n$ direct construction showing Presburger definability of reachability via Parikh images for 1-counter automata
- $M_0 * M_1$ when established for M_0 and M_1

Decidability

 $\Longrightarrow$ FO[R] is decidable

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Decidability

Goal: $M_0 * M_1$ is automatic when M_0 and M_1 are automatic

Theorem If M has *automatic rational multiplication* then M has an effectively automatic reachability structure.

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Theorem If M has *automatic rational multiplication* then M has an effectively automatic reachability structure.

M has automatic rational multiplication if

- given a finite automaton representing $R \subseteq M$
- we can compute a synchronous automaton representing

$$R^{\odot} := \{(u, v) \in M \times M \mid \exists r \in R \colon u \odot r = v\}$$

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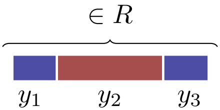
Assume $R \subseteq M_0 * M_1$ is represented by finite automaton A.



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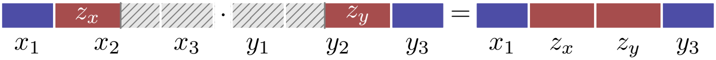

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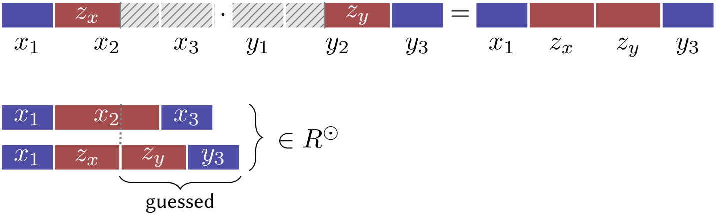
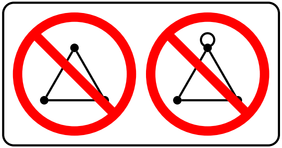
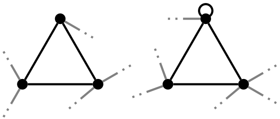


Decidability



$\Longrightarrow$ FO[R] is decidable

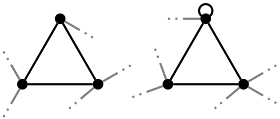
Undecidability



Γ is **not** a disjoint union of $\mathbb{B}^2$-triangle-free cliques

$\Longrightarrow$ FO[R] is undecidable

Undecidability



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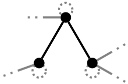
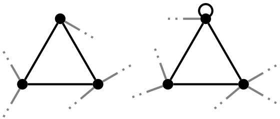
↓

Case 1: not all cliques



Submonoid $\{a, b\}^* \times \{c\}^*$

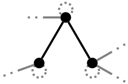
Undecidability



Γ is **not** a disjoint union of $\mathbb{B}^2$-triangle-free cliques

$\Longrightarrow$ FO[R] is undecidable

Case 1: not all cliques



Submonoid $\{a, b\}^* \times \{c\}^*$

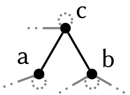
Case 2: contains a $\mathbb{B}^2$-triangle

Submonoid

$\mathbb{B}^2 \times \mathbb{B}$ or $\mathbb{B}^2 \times \mathbb{Z}$

Undecidability

Case 1: not all cliques



Submonoid $\{a, b\}^* \times \{c\}^*$

We can prove undecidability without barred symbols:

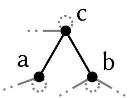
1. Use a and b as in a stack without popping.
2. Use c in a counter never decrementing.

We have two proofs:

- There is a *fixed* FO[R]-formula that cannot be checked for valence systems over $\mathbb{M}\Gamma$.
- There is a *fixed* valence system over $\mathbb{M}\Gamma$ with an undecidable first-order theory with reachability.

Undecidability

Case 1: not all cliques



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  By reducing universality of rational subsets of $\{a,b\}^* \times \{c\}^*$ which is undecidable (Sakarovitch 1992)
- There is a *fixed* valence system over $\mathbb{M}\Gamma$ with an undecidable first-order theory with reachability.

Undecidability

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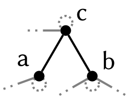

Submonoid $\{a, b\}^* \times \{c\}^*$

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- There is a *fixed* valence system over $\mathbb{M}\Gamma$ with an undecidable first-order theory with reachability.
  By reducing a variant of PCP

Undecidability

Case 2: contains a $\mathbb{B}^2$-triangle

Submonoid
$\mathbb{B}^2 \times \mathbb{B}$ or $\mathbb{B}^2 \times \mathbb{Z}$

We can use
the submonoid $\mathbb{B}^2 \times \mathbb{N}$

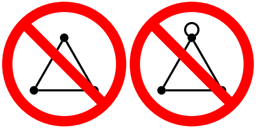
We can prove undecidability by using:

1. Two partially blind counters
2. A positive counter that we can only increment

The proof is by showing that there is a *fixed* valence automaton $A_{\mathbb{N}}$ on which $(\mathbb{N}, +, \cdot)$ can be interpreted:

- The Σ_1 fragment of arithmetic with addition and multiplication is undecidable (Matiyasevich 1993) $\Longrightarrow$ Σ_2 over $\mathbb{M}\Gamma$ is undecidable.
- Key trick:
  - squaring is enough $(a+b)^2 = a^2 + 2ab + b^2$
  - implement weak squaring by using $n^2 = \sum_{i=0}^{n-1} 2i + 1$

Conclusions

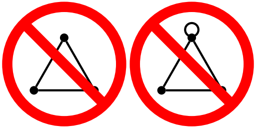


Γ is a disjoint union of $\mathbb{B}^2$-triangle-free cliques

iff

FO[R] for valence systems over $\mathbb{M}\Gamma$ is decidable

Conclusions



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iff

FO[R] for valence systems over $\mathbb{M}\Gamma$ is decidable

As an application, undecidability of FO[R] on 3-dimension VASS is a special case.

Thank you!